2024 BC #6
(no calculator)

(a)

At $x = 6$, $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)6^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}$

Comparing the series to harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges (p-series, $p = 1$)

and $\dfrac{n+1}{n^2} > \dfrac{1}{n}$ for $n \geq 1$. So the series also $\boxed{\text{diverges}}$ by the Direct Comparison Test.

OR: Using the Limit Comparison Test:

Since both series are positive, $\displaystyle\lim_{x\to\infty} \left(\frac{\frac{n+1}{n^2}}{\frac{1}{n}} \right) = \lim_{x\to\infty} \left(\frac{n+1}{n} \right) = 1 > 0$ and finite

So both series diverge by the Limit Comparison Test.

(b)

$f(-3) = \displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2} \left(-\frac{1}{2} \right)^n$ is an alternating series that converges

since the radius of convergence is 6 and $-6 < -3 < 6$, then $f(-3)$ must converge.

$\left| f(-3) - S_3 \right| = \text{Error} < \left| a_4 \right|$

$$\left| a_4 \right| = \left| \frac{4+1}{4^2} \left(-\frac{1}{2} \right)^4 \right| = \frac{5}{256} < \frac{5}{250} = \frac{1}{50}$$

(c)

$$f(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{(n+1)}{n^2 6^n} x^n \quad \Rightarrow \quad f'(x) = \sum_{n=1}^{\infty} \left(\frac{(n+1)}{n^2 6^n} \right) n x^{n-1} = \boxed{\sum_{n=1}^{\infty} \frac{(n+1)x^{n-1}}{n 6^n}}$$

Since the radius of convergence of a power series f' is the same as that of a power series f,

then the radius of convergence of f' is $\boxed{6}$.

OR: Using the ratio test to find the radius of convergence of f':

$$\lim_{n\to\infty} \left| \frac{(n+2)x^n}{(n+1)6^{n+1}} \bullet \frac{n6^n}{(n+1)x^{n-1}} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{(n+2)xn}{(n+1)^2 6} \right| < 1 \Rightarrow \lim_{n\to\infty} \left(\frac{(n+2)n}{(n+1)^2} \right) \left| \frac{x}{6} \right| < 1 \Rightarrow \left| \frac{x}{6} \right| < 1 \Rightarrow |x| < 6 \Rightarrow -6 < x < 6$$

The radius of convergence of f' is $\boxed{6}$.

(d)

$$g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$$

$$\lim_{n\to\infty} \left| \frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}} \bullet \frac{n^2 3^n}{(n+1)x^{2n}} \right| < 1$$

$$\lim_{n\to\infty} \left(\frac{(n+2)n^2}{(n+1)^3} \right) \left| \frac{x^2}{3} \right| < 1$$

$\left| \dfrac{x^2}{3} \right| < 1 \Rightarrow \left| x^2 \right| < 3 \Rightarrow |x| < \sqrt{3}$ So the radius of convergence of g is $\boxed{\sqrt{3}}$