2018 BC #6
(no calculator)

(a)

Using the Maclaurin series for $\ln(1+x)$ that was given:

$$f(x) = x\ln\left(1+\frac{x}{3}\right) = x\left[\left(\frac{x}{3}\right) - \frac{\left(\frac{x}{3}\right)^2}{2} + \frac{\left(\frac{x}{3}\right)^3}{3} - \frac{\left(\frac{x}{3}\right)^4}{4} + \cdots + (-1)^{n+1}\frac{\left(\frac{x}{3}\right)^n}{n} + \cdots\right]$$

$$= \boxed{\frac{x^2}{3} - \frac{x^3}{2\cdot 3^2} + \frac{x^4}{3\cdot 3^3} - \frac{x^5}{4\cdot 3^4} + \cdots + (-1)^{n+1}\frac{x^{n+1}}{n\cdot 3^n} + \cdots}$$

(b)

Using the ratio test to find the interval of convergence for f:

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}x^{n+2}}{(n+1)\cdot 3^{n+1}}}{\frac{(-1)^{n+1}x^{n+1}}{n\cdot 3^n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}x^{n+2}}{(n+1)\cdot 3^{n+1}}\cdot\frac{n\cdot 3^n}{(-1)^{n+1}x^{n+1}}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\left|\frac{-x}{3}\right| = \left|\frac{x}{3}\right| < 1 \text{ for convergence}$$

So, $-1 < \frac{x}{3} < 1 \Rightarrow -3 < x < 3$

Now, testing the endpoints of the interval:

$x = 3: \sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^{n+1}}{n\cdot 3^n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{n}$ which is an alternating harmonic series so it converges or it is an alternating series whose terms decrease in absolute value to 0 so it converges

$x = -3: \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-3)^{n+1}}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{3}{n}$ which is a harmonic series so it diverges or it a p-series where $p \le 1$ so it diverges

$\therefore$ the interval of convergence for f is $\boxed{-3 < x \le 3}$

(c)

Since 2 is within the interval of convergence for f,
the alternating series error bound using the 4*th*-degree Taylor polynomial
would be the absolute value of the next term, the 5*th*-degree term,
evaluated at $x = 2$. So $\left|P_4(2) - f(2)\right| < \boxed{\frac{2^5}{4\cdot 3^4}}$