2017 BC #6
(no calculator)

(a)

$$f(0)=0$$
$$f'(0)=1$$
$n=1$: $f^{(2)}(0)=-1f'(0)=-1(1)=-1$

$n=2$: $f^{(3)}(0)=-2f^{(2)}(0)=-2(-1)=2$

$n=3$: $f^{(4)}(0)=-3f^{(3)}(0)=-3(2)=-6$

$$f(x)=f(0)+f'(0)x+\frac{f^{(2)}(0)x^2}{2!}+\frac{f^{(3)}(0)x^3}{3!}+\frac{f^{(4)}(0)x^4}{4!}+\cdots$$
$$=0+1x+\frac{-1x^2}{2}+\frac{2x^3}{6}+\frac{-6x^4}{24}+\cdots$$
$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+\frac{(-1)^{n+1}x^n}{n}+\cdots \quad \text{for } n\geq 1$$

(b)

$f(1)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges because is an alternating series whose terms decrease in absolute value to 0 or because it is the alternating harmonic series.

$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right|=\sum_{n=1}^{\infty}\frac{1}{n}$ diverges because it is a p-series where $p\leq 1$ or because it is the harmonic series.

$\therefore$ the series in part (a) **converges conditionally** at $x=1$.

(c)

$$g(x)=\int_0^x f(t)\,dt=\int_0^x\left(t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\cdots\right)dt$$
$$=\left.\frac{t^2}{1\bullet 2}-\frac{t^3}{2\bullet 3}+\frac{t^4}{3\bullet 4}-\frac{t^5}{4\bullet 5}+\cdots\right|_0^x$$
$$=\frac{x^2}{1\bullet 2}-\frac{x^3}{2\bullet 3}+\frac{x^4}{3\bullet 4}-\frac{x^5}{4\bullet 5}+\cdots+\frac{(-1)^{n+1}x^{n+1}}{n(n+1)}+\cdots \quad \text{for } n\geq 1$$

Note: General term could also be $\frac{(-1)^n x^n}{(n-1)n}$ for $n\geq 2$

(d)

Since we are using $P_4\left(\frac{1}{2}\right)$ as an approximation, the alternating series error bound will be the absolute value of the next unused term (which would be the 5*th*-degree term in this case).

So $$\left|P_4\left(\frac{1}{2}\right)-g\left(\frac{1}{2}\right)\right| \leq \left|\frac{-\left(\frac{1}{2}\right)^5}{4\bullet 5}\right|=\frac{\frac{1}{32}}{20}=\frac{1}{640}<\frac{1}{500}$$